DSC – 230 (MAT-UG)

III Year B.Sc. Degree Examination, November 2008
MATHEMATICS (Paper – III)
(Directorate of Distance Education Course)

Time : 3 Hours Max. Marks : 90

Note : Answer any SIX full questions from the following.

PART – A

1. a) i) Prove that every quotient group of an abelian group is abelian. 2

   ii) If $f : (R, +) \to (R^+, \cdot)$ is a mapping defined by $f(x) = x^2$, $\forall\, x \in R$, find whether f is a homomorphism or not. 2

   b) If $f : G \to G^1$ is a homomorphism of groups then prove that the kerf is a normal subgroup of G. 5

   c) Prove that the mapping $f : G \to G$, where G is a group, defined by $f(a) = a^{-1}$, $\forall\, a \in G$ is an automorphism iff G is abelian. 6

2. a) i) Is Z_7 a field ? why ? 2

   ii) Prove that every subgroup of (z, +) is a subring of the ring (z, + , ·) of integers. 2

   b) Prove that the union of two ideals of a ring R is an ideal of R iff one is contained in the other. 5

   c) Prove that every homomorphic image of a ring R is isomorphic to the quotient ring. 6

3. a) i) Prove that $9 + 4\sqrt{5}$ is a unit in $z\left(\sqrt{5}\right)$. 2

   ii) Show that the polynomial $x^2 + 1$ is reducible over the field of complex numbers. 2

   b) Find the g.c.d of $f(x) = 4x^3 - 12x^2 - 15x - 4$ and $g(x) = 12x^2 - 24x - 15$ over Q (x) and express it as a linear combination of f (x) and g (x). 5

   c) Test the polynomial $5x^3 + 8x^2 + 6x - 4$ for rational root. 6

P.T.O.

4. a) i) Prove that the intersection of two normal subgroups of a group is a normal subgroup of G. **2**

ii) Is every integral domain always a field ? Give an example. **2**

b) Prove that a normal subgroup N of a group G is a normal subgroup G iff the product of any two right cosets of N in G is again a right coset of N in G. **5**

c) If $x^2 = x$, $\forall x \in R$, where R is a ring, then prove that R is commutative. **6**

PART – B

5. a) i) In any vector space V, prove that $0\alpha = 0$ **2**

ii) Show that a subset $S = \{(x_1, x_2, x_3) | 4x_1 + 7x_2 = 0\}$ is a subspace of V_3 (R). **2**

b) Prove that a non empty subset S of a vector space over the field F is a subspace of V iff $C_1\alpha + C_2\beta \in S$, $\forall \alpha, \beta \in S$ and $C_1, C_2 \in F$. **5**

c) Construct an addition table for $V_2(Z_2)$ and list its subspaces. **6**

6. a) i) Prove that the vectors (1, 3, 2), (2, 5, 3) and (6, 16, 10) are linearly dependent in V_3 (Q). **2**

ii) Prove that a set containing a zero vector is a linearly dependent set. **2**

b) If β is not in the subspace S but, is in the subspace spanned by S and α, then prove that α is in the subspace spanned by S and β. **5**

c) If n vectors span a vector space V over the field F and r vectors of V are linearly independent then prove that $n \geq r$. **6**

7. a) i) Verify whether the transformation **2**

$T: V_1(R) \rightarrow V_3(R)$ defined by

$T(x) = (x, 2x^2, x^3)$ is linear or not

ii) Find the matrix of the linear transformation

$T: V_3(R) \rightarrow V_2(R)$ defined by $T(X, Y, Z) = (X + Y, Y + Z)$ w.r.t. standard bases. **2**

b) If $T: V \rightarrow W$ is a non-singular linear map, then prove that $T^{-1}: W \rightarrow V$ is linear and bijective. **5**

4. a) i) Prove that the intersection of two normal subgroups of a group is a normal subgroup of G. 2

ii) Is every integral domain always a field ? Give an example. 2

b) Prove that a normal subgroup N of a group G is a normal subgroup G iff the product of any two right cosets of N in G is again a right coset of N in G. 5

c) If $x^2 = x, \forall x \in R$, where R is a ring, then prove that R is commutative. 6

PART – B

5. a) i) In any vector space V, prove that $0\alpha = 0$. 2

ii) Show that a subset $S = \{(x_1, x_2, x_3) \mid 4x_1 + 7x_2 = 0\}$ is a subspace of V_3 (R). 2

b) Prove that a non empty subset S of a vector space over the field F is a subspace of V iff $C_1\alpha + C_2\beta \in S, \forall \alpha, \beta \in S$ and $C_1, C_2 \in F$. 5

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